SPARSE BOUNDS FOR MAXIMAL ROUGH SINGULAR INTEGRALS VIA THE FOURIER TRANSFORM

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Abstract. We prove a quantified sparse bound for the maximal truncations of convolution-type singular integrals with suitable Fourier decay of the kernel. Our result extends the sparse domination principle by Conde-Alonso, Culiuc, Ou and the first author to the maximally truncated case, and covers the rough homogeneous singular integrals T_Ω on $\mathbb{R}^d$ with bounded angular part Ω having vanishing integral on the sphere. Among several consequences, we obtain new quantitative weighted norm inequalities for the maximal truncation of T_Ω, extending a result by Roncal, Tapiola and the second author.

A convex-body valued version of the sparse bound is also deduced and employed towards novel matrix-weighted norm inequalities for the maximal truncations of T_Ω. Our result is quantitative, but even the qualitative statement is new, and the present approach via sparse domination is the only one currently known for the matrix weighted bounds of this class of operators.

1. Introduction and main results

Let $\eta \in (0,1)$. A countable collection $\mathcal{S}$ of cubes of $\mathbb{R}^d$ is said to be η-sparse if there exist measurable sets $\{E_I : I \in \mathcal{S}\}$ such that

$$E_I \subset I,\ |E_I| \geq \eta |I|, \qquad I, J \in \mathcal{S}, I \neq J \implies E_I \cap E_J = \varnothing.$$

Let T be a sublinear operator mapping the space $L^\infty_0(\mathbb{R}^d)$ of complex-valued, bounded and compactly supported functions on $\mathbb{R}^d$ into locally integrable functions. We say that T has the sparse (p_1, p_2) bound [10] if there exists a constant $C > 0$ such that for all $f_1, f_2 \in L^\infty_0(\mathbb{R}^d)$ we may find a $\frac{1}{2}$-sparse collection $\mathcal{S} = \mathcal{S}(f_1, f_2)$ such that

$$|\langle T f_1, f_2 \rangle| \leq C \sum_{Q \in \mathcal{S}} |Q| \prod_{j=1}^{2} \langle f_j \rangle_{p_j, Q}$$

in which case we denote by $\|T\|_{(p_1,p_2),\mathsf{sparse}}$ the least such constant C. As customary,

$$\langle f \rangle_{p,Q} = \frac{\|f \mathbf{1}_Q\|_p}{|Q|^{\frac{1}{p}}}, \qquad p \in (0, \infty].$$

Estimating the sparse norm(s) of a sublinear or multisublinear operator entails a sharp control over the behavior of such operator in weighted L^p-spaces; this theme has been recently pursued

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by several authors, see for instance [1, 8, 18, 20, 21, 33]. This sharp control is exemplified in the following proposition, which is a collection of known facts from the indicated references.

Proposition 1.1. *Let T be a sublinear operator on $\mathbb{R}^d$ mapping $L^\infty_0(\mathbb{R}^d)$ to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then the following hold.*

1. [7, Appendix B] *Let $1 \le p_1, p_2 < \infty$. There is an absolute constant $C_{p_2} > 0$ such that*
$$\|T : L^{p_1}(\mathbb{R}^d) \to L^{p_1,\infty}(\mathbb{R}^d)\| \le C_{p_2} \|T\|_{(p_1,p_2),\mathsf{sparse}}$$
2. [12, Proposition 4.1] *If*
$$\Psi(t) := \|T\|_{\left(1+\frac{1}{t},1+\frac{1}{t}\right),\mathsf{sparse}} < \infty \qquad \forall t > 1, \tag{1.1}$$
then there is an absolute constant $C > 0$ such that
$$\|T\|_{L^2(w,\mathbb{R}^d)} \le C[w]_{A_2} \Psi\left(C[w]_{A_2}\right).$$
In particular,
$$\sup_{t>1} \Psi(t) < \infty \implies \|T\|_{L^2(w,\mathbb{R}^d)} \le C[w]_{A_2}.$$

In this article, we are concerned with the sparse norms (1.1) of a class of convolution-type singular integrals whose systematic study dates back to the celebrated works by Christ [4], Christ-Rubio de Francia [6], and Duoandikoetxea-Rubio de Francia [13], admitting a decomposition with good decay properties of the Fourier transform. To wit, let $\{K_s : \mathbb{R}^d \to \mathbb{C}, s \in \mathbb{Z}\}$ be a sequence of (smooth) functions with the properties that
$$\begin{aligned} &\operatorname{supp} K_s \subset A_s := \{x \in \mathbb{R}^d : 2^{s-4} < |x|_\infty < 2^{s-2}\}, \\ &\sup_{s\in\mathbb{Z}} 2^{sd}\|K_s\|_\infty \le 1, \\ &\sup_{s\in\mathbb{Z}} \sup_{\xi\in\mathbb{R}^d} \max\{|2^s\xi|^\alpha, |2^s\xi|^{-\alpha}\}|\widehat{K_s}(\xi)| \le 1, \end{aligned} \tag{1.2}$$
for some $\alpha > 0$. We consider truncated singular integrals of the type
$$Tf(x,t_1,t_2) = \sum_{t_1<s\le t_2} K_s * f(x), \quad t_1,t_2 \in \mathbb{Z},$$
and their maximal version
$$T_\star f(x) := \sup_{t_1\le t_2} |Tf(x,t_1,t_2)|. \tag{1.3}$$

Theorem A. *Let $T_\star$ be the maximal truncated singular integral defined in (1.3), under assumptions (1.2) on $\{K_s : s \in \mathbb{Z}\}$. Then*
$$\sup_{0<\varepsilon<1} \varepsilon\|T_\star\|_{(1+\varepsilon,1+\varepsilon),\mathsf{sparse}} \lesssim 1.$$
The implicit constant depends on dimension d only and is in particular uniform over families $\{K_s\}$ satisfying (1.2).

Theorem A entails immediately a variety of novel corollaries involving weighted norm inequalities for the maximally truncated operators $T_\star$. In addition to, for instance, those obtained by suitably applying the points of Proposition 1.1, we also detail the quantitative estimates below, whose proof will be given in Section 7.

Theorem B. *Let T be a sublinear operator satisfying the sparse bound (1.1) with $\Psi(t) \leq Ct$.*

1. *Let $1 < p < \infty$, $w \in A_p$ be a Muckenhoupt weight and $\sigma = w^{-\frac{1}{p-1}}$ be its $A_{p'}$ Muckenhoupt dual. We have the estimate*
$$\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p}}([w]_{A_p}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \max\{[\sigma]_{A_\infty}, [w]_{A_\infty}\}$$
with implicit constant possibly depending on p and dimension d; in particular,
$$\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}. \tag{1.4}$$
2. *The Fefferman-Stein type inequality*
$$\|Tf\|_{L^p(w)} \lesssim p^2(p')^{\frac{1}{p}}(r')^{1+\frac{1}{p'}}\|f\|_{L^p(M_r w)}, \qquad 1 < r < p < \infty$$
holds with implicit constant possibly depending on d only.
3. *The A_q-A_∞ estimate*
$$\|Tf\|_{L^p(w)} \lesssim [w]_{A_q}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}\|f\|_{L^p(w)}, \tag{1.5}$$
holds for $1 \leq q < p < \infty$ and $w \in A_q$, with implicit constant possibly depending on p, q and d only.
4. *The following Coifman-Fefferman type inequality*
$$\|Tf\|_{L^p(w)} \lesssim \frac{[w]_{A_\infty}^2}{\varepsilon}\|M_{1+\varepsilon}f\|_{L^p(w)}, \qquad 0 < p < \infty$$
holds for all $\varepsilon > 0$ with implicit constant possibly depending on p and d only.

Remark 1.2. Take $\Omega : S^{d-1} \to \mathbb{C}$ with $\|\Omega\|_\infty \leq 1$ and having vanishing integral on S^{d-1}, and consider the associated truncated integrals and their maximal function
$$T_{\Omega,\delta}f(x) := \int_{\delta<|t|<\frac{1}{\delta}} f(x-t)\frac{\Omega(t/|t|)}{|t|^d}\,\mathrm{d}t, \qquad T_{\Omega,\star}f(x) := \sup_{\delta>0}\left|T_{\Omega,\delta}f(x)\right|, \qquad x \in \mathbb{R}^d. \tag{1.6}$$
It is well known– for instance, see the recent contribution [16, Section 3]– that
$$T_{\Omega,\star}f(x) \lesssim \mathrm{M}f(x) + T_\star f(x), \qquad x \in \mathbb{R}^d$$
with $T_\star$ being defined as in (1.3) for a suitable choice of $\{K_s : s \in \mathbb{Z}\}$ satisfying (1.2) with $\alpha = \frac{1}{d}$. As $\|\mathrm{M}\|_{(1,1),\mathsf{sparse}} \lesssim 1$, a corollary of Theorem A is that
$$\|T_{\Omega,\star}\|_{(1+\varepsilon,1+\varepsilon),\mathsf{sparse}} \lesssim \frac{1}{\varepsilon} \tag{1.7}$$
as well. The main result of [7] is the stronger control
$$\sup_{\delta>0}\|T_{\Omega,\delta}\|_{(1,1+\varepsilon),\mathsf{sparse}} \lesssim \frac{1}{\varepsilon}. \tag{1.8}$$
The above estimate, in particular, is stronger than the uniform weak type $(1,1)$ for the operators $T_{\Omega,\delta}$, a result originally due to Seeger [31]. As the weak type $(1,1)$ of $T_{\Omega,\star}$ under no additional smoothness assumption on Ω is a difficult open question, estimating the $(1,1+\varepsilon)$ sparse norm of $T_{\Omega,\star}$ as in (1.8) seems out of reach.

The study of sharp weighted norm inequalities for $T_{\Omega,\delta}$ (the uniformity in δ is of course relevant here) was initiated in the recent article [16] by Hytönen, Roncal and Tapiola. Improved quantifications have been obtained in [7] as a consequence of the domination result (1.8), and further weighted estimates– including a Coifman-Fefferman type inequality, that is a norm control of $T_{\Omega,\delta}$ by M on all $L^p(w)$, $0 < p < \infty$ when $w \in A_\infty$– have been later derived from (1.8) in the recent preprint by the third named author, Pérez, Roncal and Rivera-Rios [28].

Although (1.7) is a bit weaker than (1.8), we see from comparison of (1.4) from Theorem B with the results of [7, 28] that the quantification of the $L^2(w)$-norm dependence on $[w]_{A_2}$ entailed by the two estimates is the same– quadratic; on the contrary, for $p \neq 2$, (1.8) yields the better estimate $\|T_{\Omega,\delta}\|_{L^p(w)} \lesssim [w]_{A_p}^{p'}$. We also observe that the proof of the mixed estimate (1.5) actually yields the following estimate for the non-maximally truncated operators, improving the previous estimate given in [28]

$$\|T_{\Omega,\delta} f\|_{L^p(w)} \lesssim [w]_{A_q}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \|f\|_{L^p(w)}.$$

Finally, we emphasize that (1.7) also yields a precise dependence on p of the unweighted L^p operator norms. Namely, from the sparse domination, we get

$$\|T_{\Omega,\star}\|_{L^p(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} \lesssim \max\{p,p'\}, \qquad \|T_{\Omega,\star}\|_{L^p(\mathbb{R}^d)} \lesssim pp' \max\{p,p'\} \tag{1.9}$$

with absolute dimensional implicit constant, which improves on the implicit constants in [13]. Moreover, we note that the main result of [29] implies that if (1.9) is sharp, then our quantitative weighted estimate (1.4) is also sharp.

Remark 1.3. We note that Theorem B holds with T being the commutator of a Calderón-Zygmund operator with a *BMO* symbol. This follows by comparing Theorem A with the sparse domination formula for commutators from [24], with the help of the John-Nirenberg inequality.

1.4. **Matrix weighted estimates for vector valued rough singular integrals.** Let $(e_\ell)_{\ell=1}^n$, $\langle \cdot,\cdot \rangle_{\mathbb{F}^n}$ and $|\cdot|_{\mathbb{F}^n}$ be the canonical basis, scalar product and norm on $\mathbb{F}^n$ over $\mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A recent trend in Harmonic Analysis– see, among others, [2, 3, 9, 14, 30]– is the study of quantitative matrix weighted norm inequalities for the canonical extension of the (integral) linear operator T

$$\langle Tf(x), e_\ell \rangle_{\mathbb{F}^n} := \langle T \otimes \mathrm{Id}_{\mathbb{F}_n} f(x), e_\ell \rangle_{\mathbb{F}^n} = T(\langle f, e_\ell \rangle_{\mathbb{F}^n})(x), \qquad x \in \mathbb{R}^d$$

to $\mathbb{F}^n$-valued functions f. In Section 6 of this paper, we introduce an L^p, $p > 1$, version of the convex body averages first brought into the sparse domination context by Nazarov, Petermichl, Treil and Volberg [30], and use them to produce a vector valued version of Theorem A. As a corollary, we obtain quantitative matrix weighted estimates for the maximal truncated vector valued extension of the rough singular integrals $T_{\Omega,\delta}$ from (1.6). In fact, the next corollary is a special case of the more precise Theorem E from Section 6.

Corollary E.1. *Let W be a positive semidefinite and locally integrable $\mathcal{L}(\mathbb{F}^n)$-valued function on $\mathbb{R}^d$ and $T_{\Omega,\delta}$ be as in (1.6). Then*

$$\left\| \sup_{\delta>0} \left| W^{\frac{1}{2}} T_{\Omega,\delta} f \right|_{\mathbb{F}^n} \right\|_{L^2(\mathbb{R}^d)} \lesssim [W]_{A_2}^{\frac{5}{2}} \left\| \left| W^{\frac{1}{2}} f \right|_{\mathbb{F}^n} \right\|_{L^2(\mathbb{R}^d)} \tag{1.10}$$

with implicit constant depending on d, n only, where the matrix A_2 constant *is given by*

$$[W]_{A_2} := \sup_{Q \text{ cube of } \mathbb{R}^d} \left\| \left(\frac{1}{|Q|} \int_Q W(x)\,\mathrm{d}x \right)^{\frac{1}{2}} \left(\frac{1}{|Q|} \int_Q W^{-1}(x)\,\mathrm{d}x \right)^{\frac{1}{2}} \right\|_{\mathcal{L}(\mathbb{F}^n)}^2 .$$

As the left hand side of (1.10) dominates the matrix weighted norm of the vector valued maximal operator first studied by Christ and Goldberg in [5], the finiteness of $[W]_{A_2}$ is actually necessary for the estimate to hold. To the best of the authors' knowledge, Theorem E has no predecessors, in the sense that no matrix weighted norm inequalities for vector rough singular integrals were known before, even in qualitative form. At this time we are unable to assess whether the power $\frac{5}{2}$ appearing in (1.10) is optimal. For comparison, if the angular part Ω is Hölder continuous, the currently best known result [30] is that (1.10) holds with power $\frac{3}{2}$; see also [9].

1.5. **Strategy of proof of the main results.** We will obtain Theorem A by an application of an abstract sparse domination principle, Theorem C from Section 3, which is a modification of [7, Theorem C]. At the core of our approach lies a special configuration of stopping cubes, the so-called *stopping collections* $\mathcal{Q}$, and their related atomic spaces. The necessary definitions, together with a useful interpolation principle for the atomic spaces, appear in Section 2. In essence, Theorem C can be summarized by the inequality

$$\|T_\star\|_{(p_1,p_2),\mathsf{sparse}} \lesssim \|T_\star\|_{\mathcal{L}(L^2(\mathbb{R}^d))} + \sup_{\mathcal{Q},t_1,t_2} \left(\|Q_{t_1}^{t_2}\|_{\dot{\mathcal{X}}_{p_1}\times\mathcal{Y}_{p_2}} + \|Q_{t_1}^{t_2}\|_{\mathcal{Y}_\infty\times\dot{\mathcal{X}}_{p_2}} \right)$$

where the supremum is taken over all stopping collections $\mathcal{Q}$ and all measurable linearizations of the truncation parameters t_1, t_2, and $Q_{t_1}^{t_2}$ are suitably adapted localizations of (the adjoint form to the linearized versions of) $T_\star$. In Section 4, we prove the required uniform estimates for the localizations $Q_{t_1}^{t_2}$ coming from Dini-smooth kernels. The proof of Theorem A is given in Section 5, relying upon the estimates of Section 4 and the Littlewood-Paley decomposition of the convolution kernels (1.2) whose first appearance dates back to [13].

Remark 1.6. We remark that while this article was being finalized, an alternative proof of (1.8) was given by Lerner [22]. It is of interest whether the strategy of [22], relying on bumped bilinear grand local maximal functions, can be applied towards estimate (1.7) as well.

Notation. With $q' = \frac{q}{q-1}$ we indicate the Lebesgue dual exponent to $q \in (1,\infty)$, with the usual extension $1' = \infty$, $\infty' = 1$. The center and the (dyadic) scale of a cube $Q \in \mathbb{R}^d$ will be denoted by c_Q and s_Q respectively, so that $|Q| = 2^{s_Q d}$. We use the notation

$$\mathrm{M}_p(f)(x) = \sup_{Q\subset\mathbb{R}^d} \langle f \rangle_{p,Q} \mathbf{1}_Q(x)$$

for the p-Hardy Littlewood maximal function and write M in place of M_1. Unless otherwise specified, the almost inequality signs $\lesssim$ imply absolute dimensional constants which may be different at each occurrence.

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2. Stopping collections and interpolation in localized spaces

The notion of stopping collection $\mathcal{Q}$ with the top (dyadic) cube Q has been introduced in [7, Section 2]: we proceed by recalling the relevant definitions. A *stopping collection* $\mathcal{Q}$ with top Q is a collection of pairwise disjoint dyadic cubes contained in $3Q$ and satisfying suitable Whitney type properties. More precisely,

$$\bigcup_{L\in \mathsf{c}\mathcal{Q}} 9L \subset \mathsf{sh}\mathcal{Q} := \bigcup_{L\in\mathcal{Q}} L \subset 3Q, \qquad \mathsf{c}\mathcal{Q} := \{L\in\mathcal{Q} : 3L\cap 2Q \neq \emptyset\}; \tag{2.1}$$

$$L, L' \in \mathcal{Q},\ L\cap L' \neq \emptyset \implies L = L'; \tag{2.2}$$

$$L\in\mathcal{Q}, L'\in N(L) \implies |s_L - s_{L'}| \le 8, \qquad N(L) := \{L'\in\mathcal{Q} : 3L\cap 3L' \neq \emptyset\}. \tag{2.3}$$

A consequence of (2.3) is that the cardinality of $N(L)$ is bounded by an absolute constant.

We proceed with the definition of the localized spaces $\mathcal{Y}_p(\mathcal{Q}), \mathcal{X}_p(\mathcal{Q}), \dot{\mathcal{X}}_p(\mathcal{Q})$, whose first appearance is in [7, Section 2]. The space $\mathcal{Y}_p(\mathcal{Q})$ is the subspace of $L^p(\mathbb{R}^d)$ of functions satisfying

$$\begin{aligned} &\operatorname{supp} f \subset 3Q, \\ &\infty > \|f\|_{\mathcal{Y}_p(\mathcal{Q})} := \begin{cases} \max\left\{ \left\| f\mathbf{1}_{\mathbb{R}^d\setminus \mathsf{sh}\mathcal{Q}}\right\|_\infty, \sup_{L\in\mathcal{Q}} \inf_{x\in\widehat{L}} \mathrm{M}_p f(x)\right\} & p<\infty \\ \|f\|_\infty & p=\infty \end{cases} \end{aligned} \tag{2.4}$$

where $\widehat{L}$ stands for the (non-dyadic) 2^5-fold dilate of L, and that $\mathcal{X}_p(\mathcal{Q})$ is the subspace of $\mathcal{Y}_p(\mathcal{Q})$ of functions satisfying

$$b = \sum_{L\in\mathcal{Q}} b_L, \qquad \operatorname{supp} b_L \subset L. \tag{2.5}$$

Finally, we write $b\in\dot{\mathcal{X}}_p(\mathcal{Q})$ if $b\in\mathcal{X}_p(\mathcal{Q})$ and each b_L has mean zero. We will omit $(\mathcal{Q})$ from the subscript of the norms whenever the stopping collection $\mathcal{Q}$ is clear from context.

There is a natural interpolation procedure involving the $\mathcal{Y}_p$-spaces. We do not strive for the most general result but restrict ourselves to proving a significant example, which is also of use to us in the proof of Theorem A.

Proposition 2.1. *Let B be a bisublinear form and A_1, A_2 be positive constants such that the estimates*

$$|B(b,f)| \le A_1\|b_1\|_{\dot{\mathcal{X}}_1(\mathcal{Q})}\|f\|_{\mathcal{Y}_1(\mathcal{Q})}, \qquad |B(g_1,g_2)| \le A_2\|g_1\|_{\mathcal{Y}_2(\mathcal{Q})}\|g_2\|_{\mathcal{Y}_2(\mathcal{Q})}$$

hold true. Then for all $0<\varepsilon<1$

$$|B(f_1,f_2)| \lesssim (A_1)^{1-\varepsilon}(A_2)^{\varepsilon}\|f_1\|_{\dot{\mathcal{X}}_p(\mathcal{Q})}\|f_2\|_{\mathcal{Y}_p(\mathcal{Q})}, \qquad p = \frac{2}{2-\varepsilon}.$$

Proof. We may assume $A_2 < A_1$, otherwise there is nothing to prove. We are allowed to normalize $A_1 = 1$. Fixing now $0<\varepsilon<1$, so that $1<p<2$, it will suffice to prove the estimate

$$|B(f_1,f_2)| \lesssim (A_2)^{\varepsilon} \tag{2.6}$$

for each pair $f_1\in\dot{\mathcal{X}}_p(\mathcal{Q})$, $f_2\in\mathcal{Y}_p(\mathcal{Q})$ with $\|f_1\|_{\dot{\mathcal{X}}_p} = \|f_2\|_{\mathcal{Y}_p} = 1$ with implied constant depending on dimension only. Let $\lambda\ge 1$ to be chosen later. Using the notation $f_{>\lambda} := f\mathbf{1}_{|f|>\lambda}$, we introduce the decompositions

$$f_1 = g_1 + b_1, \quad b_1 := \sum_{Q\in\mathcal{Q}} \left((f_1)_{>\lambda} - \frac{1}{|Q|}\int_Q (f_1)_{>\lambda} \right) \mathbf{1}_Q, \qquad f_2 = g_2 + b_2, \quad b_2 := (f_2)_{>\lambda}$$

which verify the properties

$$g_1 \in \dot{\mathcal{X}}_2(\mathcal{Q}),\ \|g_1\|_{\dot{\mathcal{X}}_p} \lesssim 1,\ \|g_1\|_{\dot{\mathcal{X}}_2} \lesssim \lambda^{1-\frac{p}{2}}, \qquad b_1 \in \dot{\mathcal{X}}_1(\mathcal{Q}),\ \|b_1\|_{\dot{\mathcal{X}}_1} \lesssim \lambda^{1-p}$$
$$\|g_2\|_{\dot{\mathcal{X}}_2} \lesssim \lambda^{1-\frac{p}{2}}, \qquad \|b_2\|_{\mathcal{X}_1} \lesssim \lambda^{1-p}.$$

We have used that b_1 is supported on the union of the cubes $Q \in \mathcal{Q}$ and has mean zero on each Q, and therefore g_1 has the same property, given that $f_1 \in \dot{\mathcal{X}}_p(\mathcal{Q})$. Therefore

$$\begin{aligned}
|B(f_1, f_2)| &\leq |B(b_1, b_2)| + |B(b_1, g_2)| + |B(g_1, b_2)| + |B(g_1, g_2)| \\
&\leq \|b_1\|_{\dot{\mathcal{X}}_1}\|b_2\|_{\mathcal{Y}_1} + \|b_1\|_{\dot{\mathcal{X}}_1}\|g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\dot{\mathcal{X}}_1}\|b_2\|_{\mathcal{Y}_1} + A_2\|g_1\|_{\mathcal{Y}_2}\|g_2\|_{\mathcal{Y}_2} \\
&\lesssim \lambda^{2-2p} + 2\lambda^{1-p} + A_2\lambda^{2-p} \lesssim \lambda^{2-2p}(1 + A_2\lambda^p)
\end{aligned}$$

which yields (2.6) with the choice $\lambda = A_2^{-\frac{1}{p}}$. □

3. A sparse domination principle for maximal truncations

We consider families of functions $[K] = \{K_s : s \in \mathbb{Z}\}$ satisfying

$$\begin{aligned}
&\operatorname{supp} K_s \subset \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| < 2^s\right\}, \\
&|||[K]||| := \sup_{s\in\mathbb{Z}} 2^{sd} \sup_{x\in\mathbb{R}^d} (\|K_s(x, \cdot)\|_\infty + \|K_s(\cdot, x)\|_\infty) < \infty
\end{aligned} \tag{3.1}$$

and associate to them the linear operators

$$T[K]f(x, t_1, t_2) := \sum_{t_1<s\leq t_2} \int_{\mathbb{R}^d} K_s(x, y)f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d,\ t_1, t_2 \in \mathbb{Z} \tag{3.2}$$

and their sublinear maximal versions

$$T_{\star}{}_{t_1}^{t_2}[K]f(x) := \sup_{t_1\leq\tau_1\leq\tau_2\leq t_2} |T[K]f(x, \tau_1, \tau_2)|, \qquad T_\star[K]f(x) = \sup_{t_1\leq t_2} |T[K]f(x, t_1, t_2)|.$$

We assume that there exists $1 < r < \infty$ such that

$$|||[K]|||_{r,\star} := \|T_\star[K]\|_{L^r(\mathbb{R}^d)} < \infty. \tag{3.3}$$

For pairs of bounded measurable functions $t_1, t_2 : \mathbb{R}^d \to \mathbb{Z}$, we also consider the linear operators

$$T[K]_{t_1}^{t_2} f(x) := T[K]f(x, t_1(x), t_2(x)), \qquad x \in \mathbb{R}^d. \tag{3.4}$$

Remark 3.1. From the definition (3.2), it follows that

$$t_1, t_2 \in \mathbb{Z},\ t_1 \geq t_2 \implies T[K]f(x, t_1, t_2) = 0.$$

In consequence, for the linearized versions defined in (3.4) we have

$$\operatorname{supp} T[K]_{t_1}^{t_2} f \subset \{x \in \mathbb{R}^d : t_2(x) - t_1(x) > 0\}.$$

A related word on notation: we will be using linearizations of the type $T[K]_{t_1}^{s_Q}$ and similar, where s_Q is the (dyadic) scale of a (dyadic) cube Q. With this we mean we are using the constant function equal to s_Q as our upper truncation function. Finally, we will be using the notations $t_2 \wedge s_Q$ for the linearizing function $x \mapsto \min\{t_2(x), s_Q\}$ and $t_1 \vee s_L$ for the linearizing function $x \mapsto \max\{t_1(x), s_L\}$.

Given two bounded measurable functions t_1, t_2 and a stopping collection $\mathcal{Q}$ with top Q, we define the localized truncated bilinear forms

$$\mathcal{Q}[K]_{t_1}^{t_2}(f_1, f_2) := \frac{1}{|Q|}\left[\left\langle T[K]_{t_1}^{t_2\wedge s_Q}(f_1\mathbf{1}_Q), f_2\right\rangle - \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\left\langle T[K]_{t_1}^{t_2\wedge s_L}(f_1\mathbf{1}_L), f_2\right\rangle\right]. \tag{3.5}$$

Remark 3.2. Note that we have normalized by the measure of Q, unlike the definitions in [7, Section 2]. Observe that as a consequence of the support assumptions in (3.1) and of the largest allowed scale being s_Q, we have

$$\mathcal{Q}[K]_{t_1}^{t_2}(f_1, f_2) = \mathcal{Q}[K]_{t_1}^{t_2}(f_1\mathbf{1}_Q, f_2\mathbf{1}_{3Q}).$$

Similarly we remark that $T[K]_{t_1}^{t_2\wedge s_L}(f\mathbf{1}_L)$ is supported on the set $3L \cap \{x \in \mathbb{R}^d : s_L - t_1(x) > 0\}$; see Remark 3.1.

Within the above framework, we have the following abstract theorem.

Theorem C. *Let $[K] = \{K_s : s \in \mathbb{Z}\}$ be a family of functions satisfying (3.1) and (3.3) above. Assume that there exist $1 \le p_1, p_2 < \infty$ such that*

$$\sup_{\substack{\|b\|_{\dot{\mathcal{X}}_{p_1}(\mathcal{Q})}=1\\ \|f\|_{\mathcal{Y}_{p_2}(\mathcal{Q})}=1}}\left|\mathcal{Q}[K]_{t_1}^{t_2}(b, f)\right| + \sup_{\substack{\|f\|_{\mathcal{Y}_\infty(\mathcal{Q})}=1\\ \|b\|_{\dot{\mathcal{X}}_{p_2}(\mathcal{Q})}=1}}\left|\mathcal{Q}[K]_{t_1}^{t_2}(f, b)\right| =: C_{\mathsf{L}}[K](p_1, p_2) < \infty. \tag{3.6}$$

hold uniformly over all bounded measurable functions t_1, t_2, and all stopping collections $\mathcal{Q}$. Then

$$\|T_\star[K]\|_{(p_1,p_2),\mathsf{sparse}} \lesssim \|[K]\|_{r,\star} + C_{\mathsf{L}}[K](p_1, p_2). \tag{3.7}$$

Proof. The proof follows essentially the same scheme of [7, Theorem C]; for this reason, we limit ourselves to providing an outline of the main steps.

Step 1. Auxiliary estimate. First of all, an immediate consequence of the assumptions of the Theorem is that the estimate

$$\left|\mathcal{Q}[K]_{t_1}^{t_2}(f_1, f_2)\right| \le C\mathbf{\Theta}_{[K],p_1,p_2}\|f_1\|_{\mathcal{Y}_{p_1}(\mathcal{Q})}\|f_2\|_{\mathcal{Y}_{p_2}(\mathcal{Q})} \tag{3.8}$$

where $\mathbf{\Theta}_{[K],p_1,p_2} := \|[K]\|_{r,\star} + C_{\mathsf{L}}[K](p_1, p_2)$, holds with $C > 0$ uniform over bounded measurable functions t_1, t_2. See [7, Lemma 2.7]. Therefore,

$$\left|\left\langle T[K]_{t_1}^{t_2\wedge s_Q}(f_1\mathbf{1}_Q), f_2\right\rangle\right| \le C\mathbf{\Theta}_{[K],p_1,p_2}|Q|\|f_1\|_{\mathcal{Y}_{p_1}(\mathcal{Q})}\|f_2\|_{\mathcal{Y}_{p_2}(\mathcal{Q})} + \sum_{\substack{L\in\mathcal{Q}\\ L\subset Q}}\left|\left\langle T[K]_{t_1}^{t_2\wedge s_L}(f_1\mathbf{1}_L), f_2\right\rangle\right| \tag{3.9}$$

Step 2. Initialization. The argument begins as follows. Fixing $f_j \in L^{p_j}(\mathbb{R}^d)$, $j = 1, 2$ with compact support, we may find measurable functions t_1, t_2 which are bounded above and below and a large enough dyadic cube Q_0 from one of the canonical 3^d dyadic systems such that $\operatorname{supp} f_1 \subset Q_0$, $\operatorname{supp} f_2 \subset 3Q_0$ and

$$\langle T_\star[K]f_1, |f_2|\rangle \le 2\left|\left\langle T[K]_{t_1}^{t_2\wedge s_{Q_0}}(f_1\mathbf{1}_{Q_0}), |f_2|\right\rangle\right|$$

and we clearly can replace f_2 by $|f_2|$ in what follows.

Step 3. Iterative process. Then, the argument proceeds via iteration over k of the following construction, which follows from (3.9) and the Calderón-Zygmund decomposition and is initialized by taking $\mathcal{S}_k = \{Q_0\}$ for $k = 0$. Given a disjoint collection of dyadic cubes $Q \in \mathcal{S}_k$ with the further Whitney property that (2.3) holds for $\mathcal{S}_k$ in place of $\mathcal{Q}$, there exists a further collection of disjoint dyadic cubes $L \in \mathcal{S}_{k+1}$ such that

- (2.3) for $\mathcal{S}_k$ in place of $\mathcal{Q}$ continues to hold,
- each subcollection $\mathcal{S}_{k+1}(Q) = \{L \in \mathcal{S}_{k+1} : L \subset 3Q\}$ is a stopping collection with top Q,

and for which for all $Q \in \mathcal{S}_k$ there holds

$$\begin{aligned} &\left|\left\langle T[K]_{t_1}^{t_2\wedge s_Q}(f_1\mathbf{1}_Q), f_2\right\rangle\right| \\ &\leq C|Q|\mathbf{\Theta}_{[K],p_1,p_2}\langle f_1\rangle_{p_1,3Q}\langle f_2\rangle_{p_2,3Q} + \sum_{\substack{L\in\mathcal{S}_{k+1}(Q)\\ L\subset Q}} \left|\left\langle T[K]_{t_1}^{t_2\wedge s_L}(f_1\mathbf{1}_L), f_2\right\rangle\right|. \end{aligned} \tag{3.10}$$

More precisely, $\mathcal{S}_{k+1}$ is composed by the maximal dyadic cubes L such that

$$9L \subset \bigcup_{Q\in\mathcal{S}_k} E_Q, \qquad E_Q := \left\{x \in 3Q : \max_{j=1,2} \frac{\mathrm{M}_{p_j}(f_j\mathbf{1}_{3Q})(x)}{\langle f_j\rangle_{p_j,3Q}} > C\right\} \tag{3.11}$$

for a suitably chosen absolute large dimensional constant C. This construction, as well as the Whitney property (2.3) results into

$$\left|Q \cap \bigcup_{L\in\mathcal{S}_{k+1}} L\right| = \left|Q \cap \bigcup_{Q'\in\mathcal{S}_k:Q'\in N(Q)} E_{Q'}\right| \leq \frac{1}{2}|Q| \qquad \forall Q \in \mathcal{S}_k,\ k = 0, 1, \ldots \tag{3.12}$$

guaranteeing that $\mathcal{T}_k := \cup_{\kappa=0}^k \mathcal{S}_\kappa$ is a sparse collection for all k. When $k = \bar{k}$ is such that $\inf\{s_Q : Q \in \mathcal{S}_{\bar{k}}\} < \inf t_1$, the iteration stops and the estimate

$$\langle T_\star[K]f_1, f_2\rangle \lesssim \mathbf{\Theta}_{[K],p_1,p_2} \sum_{Q\in\mathcal{T}_{\bar{k}}} |Q|\langle f_1\rangle_{p_1,3Q}\langle f_2\rangle_{p_2,3Q}$$

is reached. This completes the proof of Theorem C. □

4. Preliminary localized estimates for the truncated forms (3.5)

We begin by introducing our notation for the Dini constant of a family of kernels $[K]$ as in (3.1). We write

$$\|[K]\|_{\mathsf{Dini}} := \|[K]\| + \sum_{j=0}^{\infty} \varpi_j([K]) \tag{4.1}$$

where

$$\varpi_j([K]) := \sup_{s\in\mathbb{Z}} 2^{sd} \sup_{x\in\mathbb{R}^d} \sup_{\substack{h\in\mathbb{R}^d\\ |h|<2^{s-j-3}}} \begin{pmatrix} & \|K_s(x, x+\cdot) - K_s(x+h, x+\cdot)\|_\infty \\ + & \|K_s(x+\cdot, x) - K_s(x+\cdot, x+h)\|_\infty \end{pmatrix}$$

The estimates contained within the lemmata that follow are meant to be uniform over all measurable functions t_1, t_2 and all stopping collections $\mathcal{Q}$. The first one is an immediate consequence of the definitions: for a full proof, see [7, Lemma 2.3].

Lemma 4.1. *Let* $1 < r < \infty$. *Then*

$$\left|Q[K]_{t_1}^{t_2}(f_1, f_2)\right| \lesssim \|[K]\|_{r,\star}\|f_1\|_{\mathcal{Y}_r(Q)}\|f_2\|_{\mathcal{Y}_{r'}(Q)}$$

The second one is a variant of [7, Lemma 3.2]; we provide a full proof.

Lemma 4.2. *There holds*

$$\left|Q[K]_{t_1}^{t_2}(b, f)\right| \lesssim \|[K]\|_{\mathsf{Dini}}\|b\|_{\dot{\mathcal{X}}_1(Q)}\|f\|_{\mathcal{Y}_1(Q)}.$$

Proof. We consider the family K fixed and use the simplified notation $Q_{t_1}^{t_2}$ in place of $Q[K]_{t_1}^{t_2}$, and similarly for the truncated operators $T[K]$. By horizontal rescaling we can assume $|Q| = 1$. Let $b \in \dot{\mathcal{X}}_1$. Recalling the definition (2.5) and using bilinearity of $Q_{t_1}^{t_2}$ it suffices for each stopping cube $R \in \mathcal{Q}$ to prove that

$$\left|Q_{t_1}^{t_2}(b_R, f)\right| \lesssim \|[K]\|_{\mathsf{Dini}}\|b_R\|_1\|f\|_{\mathcal{Y}_1} \tag{4.2}$$

as $\|b_R\|_1 \lesssim |R|\|b\|_{\dot{\mathcal{X}}_1}$, and conclude by summing up over the disjoint $R \in \mathcal{Q}$, whose union is contained in $3Q$. We may further assume $R \subset Q$; otherwise $Q_{t_1}^{t_2}(b_R, f) = 0$. In addition we can assume f is positive, by repeating the same argument below with the real and imaginary, and positive and negative parts of f. Using the definition of the truncated forms (3.5) and the disjointness of $L \in \mathcal{Q}$,

$$\left|Q_{t_1}^{t_2}(b_R, f)\right| = \left|\left\langle T_{t_1}^{t_2\wedge s_Q}(b_R) - T_{t_1}^{t_2\wedge s_R}(b_R), f\right\rangle\right| = \left|\left\langle T_{t_1\vee s_R}^{t_2\wedge s_Q}(b_R), f\right\rangle\right| \le \left\langle T_{\star}{}_{s_R}^{s_Q} b_R, f\right\rangle.$$

Thus, if R_s denotes the cube concentric to R and whose sidelength is 2^{10+s}, using the support and Dini conditions

$$\begin{aligned}
\left|Q_{t_1}^{t_2}(b_R, f)\right| \le \left\langle T_{\star}{}_{s_R}^{s_Q} b_R, f\right\rangle &\le \sum_{s=s_R+1}^{s_Q} \int_{R_s} \left|\int_R K_s(x,y) b_R(y)\,\mathrm{d}y\right| f(x)\,\mathrm{d}x \\
&= \sum_{s=s_R+1}^{s_Q} \int_{R_s} \left|\int_R [K_s(x,y) - K_s(x,c_R)] b_R(y)\,\mathrm{d}y\right| f(x)\,\mathrm{d}x \\
&\le \|b_R\|_1 \sum_{s=s_R+1}^{s_Q} \omega_{s-s_R}([K]) 2^{-sd} \int_{R_s} f(x)\,\mathrm{d}x \\
&\lesssim \|[K]\|_{\mathsf{Dini}}\|b_R\|_1 \sup_j \langle f\rangle_{1,R_j}
\end{aligned}$$

which is bounded by the right hand side of (4.2). □

The third localized estimate is new. However, its roots lie in the well-known principle that the maximal truncations of a Dini-continuous kernel to scales larger than s do not oscillate too much on a ball of radius 2^s, see (4.7). This was recently employed, for instance, in [11, 16, 19].

Lemma 4.3. *There holds*

$$\left|Q[K]_{t_1}^{t_2}(f, b)\right| \lesssim \left(\|[K]\|_{\mathsf{Dini}} \vee \|K\|_{r,\star}\right)\|f\|_{\mathcal{Y}_\infty(Q)}\|b\|_{\mathcal{X}_1(Q)}.$$

Proof. We use similar notation as in the previous proof and again we rescale to $|Q| = 1$, and work with positive $b \in \mathcal{X}_1$. We can of course assume that $\operatorname{supp} f \subset Q$. We begin by removing an error term; namely, referring to notation (2.1), if

$$b_{\mathsf{o}} = \sum_{R \notin \mathsf{c}Q} b_R$$

then

$$\left|Q[K]_{t_1}^{t_2}(f, b_{\mathsf{o}})\right| \le \langle |Tf(\cdot, s_Q - 1, s_Q)|, b_{\mathsf{o}} \rangle \lesssim |||[K]||| \|b\|_{\mathcal{X}_1} \|f\|_{\mathcal{Y}_\infty} \tag{4.3}$$

The first inequality holds because $\mathrm{dist}(\mathrm{supp} f, \mathrm{supp}\, b_{\mathsf{o}}) > 2^{s_Q - 1}$, so at most the s_Q scale may contribute, and in particular no contribution comes from cubes $L \subsetneq Q$. The second inequality is a trivial estimate, see [7, Appendix A] for more details. Thus we may assume $b_R = 0$ whenever $R \notin \mathsf{c}Q$. We begin the main argument by fixing $R \in \mathsf{c}Q$. Then by support considerations

$$\left\langle T_{t_1}^{t_2 \wedge s_L}(f \mathbf{1}_L), b_R \right\rangle \neq 0 \implies L \in N(R).$$

Similarly,

$$\left\langle T_{t_1}^{t_2 \wedge s_R} f, b_R \right\rangle = \left\langle T_{t_1}^{t_2 \wedge s_R}(f \mathbf{1}_{\mathsf{sh}Q}), b_R \right\rangle = \sum_{\substack{L \in N(R) \\ L \subset Q}} \left\langle T_{t_1}^{t_2 \wedge s_R}(f \mathbf{1}_L), b_R \right\rangle.$$

In fact, using (2.1) we learn that $\mathrm{dist}(\mathsf{sh}Q, R) > 2^{s_R}$, whence the first equality. Therefore, subtracting and adding the last display to obtain the second equality,

$$\begin{aligned} Q_{t_1}^{t_2}(f, b_R) &= \left\langle T_{t_1}^{t_2 \wedge s_Q} f, b_R \right\rangle - \sum_{\substack{L \in N(R) \\ L \subset Q}} \left\langle T_{t_1}^{t_2 \wedge s_L}(f \mathbf{1}_L), b_R \right\rangle \\ &= \left\langle T_{t_1 \vee s_R}^{t_2 \wedge s_Q} f, b_R \right\rangle - \sum_{\substack{L \in N(R) \\ L \subset Q}} \mathrm{sign}(s_L - s_R) \left\langle T_{t_1 \vee (s_L \wedge s_R)}^{t_2 \wedge (s_L \vee s_R)}(f \mathbf{1}_L), b_R \right\rangle. \end{aligned}$$

Now, the summation in the above display is then bounded in absolute value by

$$\sum_{L \in N(R)} \left\langle T_{\star}{}_{s_L \vee s_R}^{s_L \wedge s_R}(f \mathbf{1}_L), b_R \right\rangle \lesssim |||[K]||| \sum_{L \in N(R)} \|b_R\|_1 \langle f \rangle_{1,L} \lesssim |[K]\|_{\mathsf{Dini}} \|b_R\|_1 \|f\|_{\mathcal{Y}_\infty},$$

using that $|s_L - s_R| \le 8$ whenever $L \in N(R)$. Therefore when $R \in \mathsf{c}Q$

$$\left|Q_{t_1}^{t_2}(f, b_R)\right| \le \left|\left\langle T_{t_1 \vee s_R}^{t_2 \wedge s_Q} f, b_R \right\rangle\right| + C \|[K]\|_{\mathsf{Dini}} \|b_R\|_1 \|f\|_{\mathcal{Y}_\infty} \tag{4.4}$$

with absolute constant C. Now, define the function

$$F(x) = \begin{cases} \sup\limits_{s_R \le \tau_1 \le \tau_2 \le s_Q} |Tf(x, \tau_1, \tau_2)| & x \in R \in \mathsf{c}Q \\ 0 & x \notin \bigcup_{R \in \mathsf{c}Q} R \end{cases}$$

and notice that $|T_{t_1 \vee s_R}^{t_2 \wedge s_Q} f| \le F$ on $R \in \mathsf{c}Q$. Since b is positive, using (4.3), summing (4.4) over $R \in \mathsf{c}Q$ and using that this is a pairwise disjoint collection, we obtain that

$$\begin{aligned} |Q_{t_1}^{t_2}(f, b)| &\le |Q_{t_1}^{t_2}(f, b_{\mathsf{o}})| + \sum_{R \in \mathsf{c}Q} |Q_{t_1}^{t_2}(f, b_R)| \\ &\le C \|[K]\|_{\mathsf{Dini}} \|b\|_{\mathcal{Y}_1} \|f\|_{\mathcal{Y}_\infty} + \sum_{R \in \mathsf{c}Q} \left|\left\langle T_{t_1 \vee s_R}^{t_2 \wedge s_Q} f, b_R \right\rangle\right| \\ &\le C \|[K]\|_{\mathsf{Dini}} \|b\|_{\mathcal{X}_1} \|f\|_{\mathcal{Y}_\infty} + \sum_{R \in \mathsf{c}Q} \langle F, b_R \rangle = C \|[K]\|_{\mathsf{Dini}} \|b\|_{\mathcal{X}_1} \|f\|_{\mathcal{Y}_\infty} + \langle F, b \rangle. \end{aligned} \tag{4.5}$$

Therefore, we are left with bounding $\langle F, b\rangle$. This is actually done using both the L^r estimate and the Dini cancellation condition. In fact, decompose

$$b = g + z, \qquad g = \sum_{R\in cQ} g_R := \sum_{R\in cQ} \langle b\rangle_{1,R} \mathbf{1}_R, \qquad z = \sum_{R\in cQ} z_R := \sum_{R\in cQ} (b - \langle b\rangle_{1,R})\, \mathbf{1}_R$$

so that

$$\|g\|_{\mathcal{Y}_\infty} \le \|b\|_{\mathcal{X}_1}, \qquad \|z\|_{\dot{\mathcal{X}}_1} \le 2\|b\|_{\mathcal{X}_1}$$

Then

$$\langle F, g\rangle \le \langle T_\star f, g\rangle \le \|[K]\|_{r,\star}\|f\|_r\|g\|_{r'} \le \|[K]\|_{r,\star}\|g\|_{\mathcal{Y}_\infty}\|f\|_{\mathcal{Y}_\infty} \le \|[K]\|_{r,\star}\|f\|_{\mathcal{Y}_\infty}\|b\|_{\mathcal{X}_1} \tag{4.6}$$

and we are left to control $|\langle F, z\rangle|$. We recall from [16, Lemma 2.3] the inequality

$$|Tf(x,\tau_1,\tau_2) - Tf(\xi,\tau_1,\tau_2)| \lesssim \|[K]\|_{\mathsf{Dini}} \sup_{s\ge s_R} \langle f\rangle_{1,R_s}, \qquad x,\xi\in R, \tau_2\ge\tau_1\ge s_R \tag{4.7}$$

where R_s is the cube concentric with R and sidelength 2^s, whence for suitable absolute constant C

$$F(x) \le F(\xi) + C\|[K]\|_{\mathsf{Dini}}\|f\|_{\mathcal{Y}_1}, \qquad x,\xi\in R$$

and taking averages there holds

$$\sup_{x\in R} |F(x) - \langle F\rangle_{1,R}| \lesssim \|[K]\|_{\mathsf{Dini}}\|f\|_{\mathcal{Y}_1}.$$

Finally, using the above display and the fact that each z_R has zero average and is supported on R,

$$\begin{aligned} |\langle F, z\rangle| &\le \sum_{R\in Q} |\langle F, z_R\rangle| = \sum_{R\in cQ} |\langle F - \langle F\rangle_{1,R}\mathbf{1}_R, z_R\rangle| \lesssim \|[K]\|_{\mathsf{Dini}}\|f\|_{\mathcal{Y}_1} \sum_{R\in cQ} \|z_R\|_1 \\ &\le \|[K]\|_{\mathsf{Dini}}\|f\|_{\mathcal{Y}_\infty}\|b\|_{\mathcal{X}_1} \end{aligned} \tag{4.8}$$

and collecting (4.5), (4.6) and (4.8) completes the proof of the Lemma. □

By Proposition 2.1 applied to the forms $Q_{t_1}^{t_2}[K]$, we may interpolate the bound of Lemma 4.2 with the one of Lemma 4.1 with $r = 2$. A similar but easier procedure allows to interpolate Lemma 4.3 with Lemma 4.1 with $r = 2$. We summarize the result of such interpolations in the following lemma.

Lemma 4.4. *For $0 \le \varepsilon \le 1$ and $p = \frac{2}{2-\varepsilon}$ there holds*

$$C_{\mathsf{L}}[K](p,p) \lesssim \left(\|[K]\|_{\mathsf{Dini}} \vee \|[K]\|_{2,\star}\right)^{1-\varepsilon} \left(\|[K]\|_{2,\star}\right)^{\varepsilon}$$

where $C_{\mathsf{L}}[K](p_1,p_2)$ is defined in (3.6).

Remark 4.5 (Calderón-Zygmund theory)**.** Let T be an $L^2(\mathbb{R}^d)$-bounded singular integral operator with Dini-continuous kernel K. Then its maximal truncations obey the estimate

$$T_\star f(x) := \sup_{\delta>0} \left| \int_{\delta<|h|<\frac{1}{\delta}} K(x,x+h)f(x+h)\,\mathrm{d}h \right| \lesssim \mathrm{M}f(x) + T_\star[K]f(x), \tag{4.9}$$

with the family $[K] := \{K_s : s\in\mathbb{Z}\}$ defined by

$$K_s(x,x+h) := K(x,x+h)\psi(2^{-s}h), \qquad x,h\in\mathbb{R}^d$$

where the smooth radial function ψ satisfies

$$\operatorname{supp}\psi \subset \{h \in \mathbb{R}^d : 2^{-2} < |h| < 1\}, \qquad \sum_{s\in\mathbb{Z}} \psi(2^{-s}h) = 1,\ h \neq 0.$$

We know from classical theory [32, Ch. I.7] that $\|[K]\|_{2,\star} \lesssim \|T\|_{L^2(\mathbb{R}^d)} + \|K\|_{\mathsf{Dini}}$. Therefore, in consequence of (4.9) and of the bound $\|\mathrm{M}\|_{(1,1),\mathsf{sparse}} \lesssim 1$, an application of Theorem C in conjunction with Lemmata 4.2 and 4.3 yields that

$$\|T_\star\|_{(1,1),\mathsf{sparse}} \lesssim \|T\|_{L^2(\mathbb{R}^d)} + \|K\|_{\mathsf{Dini}}.$$

This is a well-known result. The dual pointwise version was first obtained in this form in [16] quantifying the initial result of Lacey [19]; see also [23]. An extension to multilinear operators with less regular kernels was recently obtained in [27].

5. Proof of Theorem A

In this section, we will prove Theorem A by appealing to Theorem C for the family $[K] = \{(x,y) \mapsto K_s(x-y) : s \in \mathbb{Z}\}$ of (1.2). First of all, we notice that the assumption (3.1) is a direct consequence of (1.2). It is known from e.g. [13] (and our work below actually reproves this) that, with reference to (1.3)

$$\|T_\star[K]\|_{L^2(\mathbb{R}^d)} \lesssim 1,$$

which is assumption (3.3) with $r = 2$. Therefore, for an application of Theorem C with

$$p_1 = p_2 = p = \frac{2}{2-\varepsilon}, \qquad 0 < \varepsilon < 1$$

we are left with verifying the corresponding stopping estimates (3.6) hold with $C_{\mathsf{L}}[K](p,p) \lesssim \varepsilon^{-1}$. We do so by means of a Littlewood-Paley decomposition, as follows. Let φ be a smooth radial function on $\mathbb{R}^d$ with support in a sufficiently small ball containing the origin, having mean zero and such that

$$\sum_{k=-\infty}^{\infty} \widehat{\varphi_k}(\xi) = 1, \quad \forall \xi \neq 0, \qquad \varphi_k(\cdot) := 2^{-kd}\varphi(2^{-k}\cdot).$$

Also define

$$\phi_k(\cdot) := \sum_{\ell \geq k} \varphi_\ell(\cdot), \qquad K_{s,0} = K_s * \phi_s, \qquad K_{s,j} = \sum_{\ell=\Delta(j-1)+1}^{\Delta j} K_s * \varphi_{s-\ell},\ j \geq 1. \tag{5.1}$$

for some large integer Δ which will be specified during the proof. Unless otherwise specified, the implied constants appearing below are independent of Δ but may depend on $\alpha > 0$ from (1.2) and on the dimension. Note that $K_{s,j}$ are supported in $\{|x| < 2^s\}$. Define now for all $j \geq 0$

$$[K^j] = \{(x,y) \mapsto K_{s,j}(x-y) : s \in \mathbb{Z}\}$$

and note that, with unconditional convergence

$$K_s(y) = \sum_{j=0}^{\infty} K_{s,j}(y), \qquad y \in \mathbb{R}^d. \tag{5.2}$$

The following computation is carried out in [16, Section 3].

Lemma 5.1. *There holds*

$$\varpi_\ell([K^j]) \lesssim \min\{1, 2^{\Delta j-\ell}\} \tag{5.3}$$

and as a consequence $\|[K^j]\|_{\mathsf{Dini}} \lesssim 1 + \Delta j$ *for all* $j \geq 0$.

It is also well-known that

$$\sup_{t_1,t_2\in\mathbb{Z}} \left\| f \mapsto T[K^j]f(\cdot, t_1, t_2) = \sum_{t_1<s\leq t_2} K_{s,j} * f \right\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-\alpha\Delta(j-1)}; \tag{5.4}$$

however, we need a stronger estimate on the pointwise maximal truncations, which is implicit in [13].

Lemma 5.2. *There holds*

$$\|[K^0]\|_{2,\star} \lesssim 1, \qquad \|[K^j]\|_{2,\star} \lesssim 2^{-\frac{\alpha}{2}\Delta(j-1)}, \quad j \geq 1.$$

Proof. Let β be a smooth compactly supported function on $\mathbb{R}^d$ normalized to have $\langle\beta, 1\rangle = 1$, and write $\beta_s(\cdot) = 2^{-sd}\beta(2^{-s}\cdot)$. By usual arguments it suffices to estimate the $L^2(\mathbb{R}^d)$ operator norm of

$$f \mapsto \sup_{t_1\leq s\leq t_2} T[K^j]f(\cdot, s, t_2)$$

uniformly over $t_1, t_2 \in \mathbb{Z}$. We then have

$$\begin{aligned} T[K^j]f(\cdot, s, t_2) &= \beta_s * \Big(\sum_{t_1<k\leq t_2} K_k * (\phi_{k-\Delta j} - \phi_{k-\Delta(j-1)})\Big) * f \\ &\quad - \beta_s * \Big(\sum_{t_1<k\leq s} K_k * (\phi_{k-\Delta j} - \phi_{k-\Delta(j-1)})\Big) * f \\ &\quad + (\delta - \beta_s) * \Big(\sum_{s<k\leq t_2} K_k * (\phi_{k-\Delta j} - \phi_{k-\Delta(j-1)})\Big) * f =: I_{1,s} + I_{2,s} + I_{3,s}, \end{aligned} \tag{5.5}$$

For $I_{1,s}$, by (5.4) we have

$$\left\| \sup_{t_1\leq s\leq t_2} |I_{1,s}| \right\|_2 \lesssim \left\| \mathrm{M}(T[K^j]f(\cdot, t_1, t_2)) \right\|_2 \lesssim 2^{-\alpha\Delta(j-1)}\|f\|_2. \tag{5.6}$$

Next we estimate the second and third contribution in (5.5). We have, using the third assertion in (1.2), that

$$\begin{aligned} &\Big| \sum_{t_1<k\leq s} \hat{\beta}(2^s\xi)\hat{K}_k(\xi)(\hat{\phi}(2^{k-\Delta j}\xi) - \hat{\phi}(2^{k-\Delta(j-1)}\xi))\Big| \\ &\lesssim \sum_{t_1<k\leq s} \min\{1, |2^s\xi|^{-1}\} \cdot \min\{|2^k\xi|^\alpha, |2^k\xi|^{-\alpha}\} \cdot \sum_{\ell=\Delta(j-1)}^{\Delta j} \min\{|2^{k-\ell}\xi|, |2^{k-\ell}\xi|^{-1}\} \\ &\lesssim \begin{cases} 2^{-\alpha\Delta(j-1)}|2^s\xi|^{-1}, & |2^s\xi| > 1, \\ 2^{-\Delta(j-1)}|2^s\xi|, & |2^s\xi| \leq 1 \end{cases} \end{aligned}$$

A similar computation reveals

$$\Big| \sum_{k\geq s}(1 - \hat{\beta}(2^s\xi))\hat{K}_k(\xi)(\hat{\phi}(2^{k-\Delta j}\xi) - \hat{\phi}(2^{k-\Delta(j-1)}\xi))\Big| \leq 2^{-\alpha\Delta(j-1)/2} \min\{|2^s\xi|, |2^s\xi|^{-\alpha/2}\}.$$

Thus by Plancherel, for $m = 2,3$ we have

$$\left\| \sup_{t_1 \le s \le t_2} |I_{m,s}| \right\|_2 \le \left\| \Big(\sum_{s=t_1}^{t_2} |I_{m,s}|^2 \Big)^{\frac12} \right\|_2 \lesssim 2^{-\alpha\Delta(j-1)/2} \|f\|_2 \tag{5.7}$$

and the proof of the Lemma is completed by putting together (5.5)–(5.7). □

We are now ready to verify the assumptions (3.6) for the truncated forms $Q[K]_{t_1}^{t_2}$ associated to a family $[K]$ satisfying the assumptions (1.2). By virtue of Lemma 5.1 and 5.2, Lemma 4.4 applied to the families $[K^j]$ for the value $\Delta = 2\varepsilon^{-1}\alpha^{-1}$ yields that

$$C_{\mathsf{L}}[K^j](p,p) \lesssim \Big(|||[K^j]|||_{\mathsf{Dini}} \vee |[K^j]||_{2,\star} \Big)^{1-\varepsilon} \Big(|||[K^j]||_{2,\star} \Big)^{\varepsilon} \lesssim (1+\Delta j)^{1-\varepsilon} 2^{-\frac{\alpha}{2}\Delta(j-1)\varepsilon} \lesssim \varepsilon^{-1}(1+j)2^{-j}.$$

Therefore using linearity in the kernel family $[K]$ of the truncated forms $Q_{t_1}^{t_2}[K]$ and the decomposition (5.1)-(5.2)

$$C_{\mathsf{L}}[K](p,p) \le \sum_{j=0}^{\infty} C_{\mathsf{L}}[K^j](p,p) \lesssim \varepsilon^{-1} \tag{5.8}$$

which, together with the previous observations, completes the proof of Theorem A.

6. Extension to vector-valued functions

In this section, we suitably extend the abstract domination principle Theorem C to (a suitably defined) $\mathbb{F}^n$-valued extension, with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, of the singular integrals of Section 3. In fact, the $\mathbb{C}^n$-valued case can be recovered by suitable interpretation of the $\mathbb{R}^{2n}$-valued one; thus, it suffices to consider $\mathbb{F} = \mathbb{R}$.

6.1. **Convex body domination.** Our sparse domination principle for vector valued rough maximal truncations involves a generalization of the convex body domination first introduced by Nazarov, Petermichl, Treil and Volberg in [30]: we send to Remark 6.3 below for a more detailed account.

Let $1 \le p < \infty$. To each $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$ and each cube Q in $\mathbb{R}^d$, we associate the closed convex symmetric subset of $\mathbb{R}^n$

$$\langle f \rangle_{p,Q} := \left\{ \frac{1}{|Q|} \int_Q f\varphi \,\mathrm{d}x : \varphi \in \Phi_{p'}(Q) \right\} \subset \mathbb{R}^n, \tag{6.1}$$

where we used the notation $\Phi_q(Q) := \{\varphi : Q \to \mathbb{R}, \langle \varphi \rangle_{q,Q} \le 1\}$. It is easy to see that

$$\sup_{\xi \in \langle f \rangle_{p,Q}} |\xi| \le \langle |f|_{\mathbb{R}^n} \rangle_{p,Q}$$

where $\langle \cdot \rangle_{p,Q}$ on the right hand side is being interpreted in the usual fashion. A slightly less obvious fact that we will use below is recorded in the following simple lemma, which involves the notion of *John ellipsoid* of a closed convex symmetric set K. This set, which we denote by $\mathcal{E}_K$, stands for the solid ellipsoid of largest volume contained in K; in particular, the John ellipsoid of K has the property that

$$\mathcal{E}_K \subset K \subset \sqrt{n}\mathcal{E}_K \tag{6.2}$$

where, if $A \subset \mathbb{R}^n$ and $c \ge 0$, by cA we mean the set $\{ca : a \in A\}$. We also apply this notion in the degenerate case as follows: if the linear span of K is a k-dimensional subspace V of $\mathbb{R}^n$, we denote by $\mathcal{E}_K$ the solid ellipsoid of largest k-dimensional volume contained in K. In this case, (6.2) holds with $\sqrt{k}$ in place of $\sqrt{n}$, but then it also holds as stated, since $k \le n$ and $\mathcal{E}_K$ is also convex and symmetric.

Lemma 6.2. *Let $f = (f_1, \dots, f_n) \in L^p_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ and suppose that $\mathcal{E}_{\langle f \rangle_{p,Q}} = B_1$, where $B_\rho = \{a \in \mathbb{R}^n : |a|_{\mathbb{R}^n} \le \rho\}$. Then*

$$\sup_{j=1,\dots,N} \langle f_j \rangle_{p,Q} \le \sqrt{n}.$$

Proof. By definition of $\langle \cdot \rangle_{p,Q}$ for scalar functions,

$$\langle f_j \rangle_{p,Q} = \sup_{\varphi \in \Phi_{p'}(Q)} \frac{1}{|Q|} \int_Q f_j \varphi \,\mathrm{d}x = \frac{1}{|Q|} \int f_j \varphi_\star \,\mathrm{d}x$$

for a suitable $\varphi_\star \in \Phi_{p'}(Q)$. Thus $\langle f_j \rangle_{p,Q}$ is the j-th component of the vector

$$f_{\varphi_\star} = \frac{1}{|Q|} \int_Q f_j \varphi_\star \,\mathrm{d}x \in \langle f \rangle_{p,Q}.$$

By (6.2), and in consequence of the assumption, $f_{\varphi_\star} \in B_{\sqrt{n}}$, which proves the assertion. □

We now define the sparse (p_1, p_2) norm of a linear operator T mapping the space $L^\infty_0(\mathbb{R}^d; \mathbb{R}^n)$ into locally integrable, $\mathbb{R}^n$-valued functions, as the least constant $C > 0$ such that for each pair $f_1, f_2 \in L^\infty_0(\mathbb{R}^d; \mathbb{R}^n)$ we may find a $\frac{1}{2}$-sparse collection $\mathcal{S}$ such that

$$|\langle T f_1, f_2 \rangle| \le C \sum_{Q \in \mathcal{S}} |Q| \langle f_1 \rangle_{p_1,Q} \langle f_2 \rangle_{p_2,Q}. \tag{6.3}$$

We interpret the rightmost product in the above display as the right endpoint of the Minkowski product $AB = \{\langle a, b \rangle_{\mathbb{R}^n} : a \in A, b \in B\}$ of the closed convex symmetric sets $A, B \subset \mathbb{R}^n$, which is a closed symmetric interval. We use the same familiar notation $\|T\|_{(p_1,p_2),\mathsf{sparse}}$ for such norm.

Within such framework, we have the following extension of Theorem C.

Theorem D. *Let $[K] = \{K_s : s \in \mathbb{Z}\}$ be a family of real-valued functions satisfying (3.1), (3.3), and (3.6) for some $1 < r < \infty, 1 \le p_1, p_2 < \infty$. Then the $\mathbb{R}^n$-valued extension of the linearized truncations $T[K]^{t_2}_{t_1}$ defined in (3.4) admits a (p_1, p_2) sparse bound, namely*

$$\left\| T[K]^{t_2}_{t_1} \otimes \mathrm{Id}_{\mathbb{R}^n} \right\|_{(p_1,p_2),\mathsf{sparse}} \lesssim \|[K]\|_{r,\star} + C_{\mathsf{L}}[K](p_1, p_2) \tag{6.4}$$

with implicit constant possibly depending on r, p_1, p_2 and the dimensions d, n only, and in particular uniform over bounded measurable truncation functions t_1, t_2.

Remark 6.3. The sets (6.1) for $p = 1$ have been introduced in this context by Nazarov, Petermichl, Treil and Volberg [30], where sparse domination of vector valued singular integrals by the Minkowski sum of convex bodies (6.1) is employed towards matrix-weighted norm inequalities. In [9], a similar result, but in the dual form (6.3) with $p_1 = p_2 = 1$ is proved for dyadic shifts via a different iterative technique which is a basic version of the proof of Theorem C. Subsequent developments in vector valued sparse domination include the sharp estimate for the dyadic square function [14]. The usage of exponents $p > 1$ in (6.1), necessary to effectively tackle rough singular integral operators, is a novelty of this paper.

6.4. **Matrix-weighted norm inequalities.** We now detail an application of Theorem D to matrix-weighted norm inequalities for maximally truncated, rough singular integrals. In particular, Corollary E.1 from the introduction is a particular case of Theorem E below.

The classes of weights we are concerned with are the following. A pair of matrix-valued weights $W, V \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{L}(\mathbb{R}^n))$ is said to satisfy the (joint) *matrix A_2 condition* if

$$[W, V]_{A_2} := \sup_Q \left\| \sqrt{W_Q}\sqrt{V_Q} \right\|^2_{\mathcal{L}(\mathbb{R}^n)} < \infty$$

supremum being taken over all cubes $Q \subset \mathbb{R}^d$ and

$$W_Q := \frac{1}{|Q|}\int_Q W(x)\,\mathrm{d}x \in \mathcal{L}(\mathbb{R}^n).$$

We simply write $[W]_{A_2} := [W, W^{-1}]_{A_2}$. We further introduce a directional matrix A_∞ condition, namely

$$[W]_{A_\infty} := \sup_{\xi\in S^{n-1}} [\langle W\xi, \xi\rangle_{\mathbb{R}^n}]_{A_\infty} \le \sup_{\xi\in S^{n-1}} [\langle W\xi, \xi\rangle_{\mathbb{R}^n}]_{A_2} \le [W]_{A_2}.$$

where the second inequality is the content of [30, Lemma 4.3].

Theorem E. *Let $W, V \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathcal{L}(\mathbb{R}^n))$ be a pair of matrix weights, and $T_{\Omega,\delta}$ be defined by (1.6), with in particular $\|\Omega\|_\infty \le 1$. Then*

$$\left\| \sup_{\delta>0} \left| W^{\frac12} T_{\Omega,\delta}(V^{\frac12} f)\right|_{\mathbb{R}^n} \right\|_{L^2(\mathbb{R}^d)} \lesssim \max\{[W]_{A_\infty}, [V]_{A_\infty}\} \sqrt{[W,V]_{A_2}[W]_{A_\infty}[V]_{A_\infty}}\, \|f\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}\,. \tag{6.5}$$

We now explain how an application of Theorem D reduces Theorem E to a weighted square function-type estimate for convex-body valued sparse operators. First of all, fix f of unit norm in $L^2(\mathbb{R}^d; \mathbb{R}^n)$. We may then find g of unit norm in $L^2(\mathbb{R}^d; \mathbb{R}^n)$ and bounded measurable functions t_1, t_2 such that the left hand side of (6.5) is bounded by twice the sum of

$$\left| \left\langle T[K]^{t_2}_{t_1} \otimes \mathrm{Id}_{\mathbb{R}^n}(V^{\frac12} f), W^{\frac12} g \right\rangle \right| \tag{6.6}$$

and

$$\left\| x \mapsto \sup_{Q\ni x} \left\langle \left| W^{\frac12}(x) V^{\frac12} f \right|_{\mathbb{R}^n} \right\rangle_Q \right\|_{L^2(\mathbb{R}^d)} \le \left\| x \mapsto \sup_{Q\ni x} \left| W^{\frac12}(x) \langle V\rangle_Q^{\frac12} \right|_{\mathcal{L}(\mathbb{R}^n)} \left\langle \left| \langle V\rangle_Q^{-\frac12} V^{\frac12} f \right|_{\mathbb{R}^n} \right\rangle_Q \right\|_{L^2(\mathbb{R}^d)},$$

where $[K]$ is the decomposition of the kernel of T_Ω performed in [16, Section 3]. As we already remarked in the scalar valued case, such decomposition satisfies (1.2). The latter expression is (the norm of) a two-weight version of the matrix weighted maximal function of Christ and Goldberg [5]. In the one weight case when $V = W^{-1} \in A_2$, its boundedness has been proved in [5] and quantified in [17], which contains the explicit bound

$$c_{d,n}[W]_{A_2}\|f\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}$$

and the implicit improvement

$$c_{d,n}[W]^{1/2}_{A_2}[W^{-1}]^{1/2}_{A_\infty}\|f\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)},$$

where $c_{d,n}$ is a dimensional constant. A straightforward modification of the same argument, using the splitting on the right of the previous display, gives the bound

$$[W, V]^{1/2}_{A_2}[V]^{1/2}_{A_\infty}\|f\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}$$

in the two weight case. Roughly speaking, the first factor is controlled by the two weight A_2 condition and the second one by the A_∞ property of V.

By virtue of the localized estimate (5.8) for $[K]$, an application of Theorem D tells us that (6.6) is bounded by C/ε times a sparse sublinear form as in (6.3) with $p_1 = p_2 = 1+\varepsilon$, $f_1 = V^{\frac12} f$ and $f_2 = W^{\frac12} g$ for all $\varepsilon > 0$. Finally, we gather that

$$\begin{aligned}(6.7)\qquad &\left\| \sup_{\delta>0} \left| W^{\frac12} T_{\Omega,\delta}(V^{\frac12} f)\right|_{\mathbb{R}^n} \right\|_2 \\ &\lesssim \sqrt{[V,W]_{A_2} \max\{[W]_{A_\infty},[V]_{A_\infty}\}} + \inf_{\varepsilon>0} \sup_{\mathcal{S}} \frac{1}{\varepsilon} \sum_{Q\in\mathcal{S}} |Q| \langle V^{\frac12} f\rangle_{1+\varepsilon,Q} \langle W^{\frac12} g\rangle_{1+\varepsilon,Q}\end{aligned}$$

where the supremum is being taken over $\frac12$-sparse collections $\mathcal{S}$, and the proof of Theorem E is completed by the following proposition.

Proposition 6.5. *The estimate*

$$\inf_{\varepsilon>0} \frac{1}{\varepsilon} \sum_{Q\in\mathcal{S}} |Q| \langle V^{\frac12} f\rangle_{1+\varepsilon,Q} \langle W^{\frac12} g\rangle_{1+\varepsilon,Q} \lesssim \max\{[W]_{A_\infty},[V]_{A_\infty}\} \sqrt{[W,V]_{A_2}[W]_{A_\infty}[V]_{A_\infty}}$$

holds uniformly over all f, g of unit norm in $L^2(\mathbb{R}^d;\mathbb{R}^n)$ and all $\frac12$-sparse collections $\mathcal{S}$.

Proof. There is no loss in generality with assuming that the sparse collection $\mathcal{S}$ is a subset of a standard dyadic grid in $\mathbb{R}^d$, and we do so. Fix $\varepsilon > 0$. By standard reductions, we have that

$$(6.8)\qquad \sup_{\|f_j\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}=1} \sum_{Q\in\mathcal{S}} |Q| \langle V^{\frac12} f_1\rangle_{1+\varepsilon,Q} \langle W^{\frac12} f_2\rangle_{1+\varepsilon,Q} \lesssim \sqrt{[W,V]_{A_2}} \sup_{\|f_j\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}=1} \|S_{V,\varepsilon} f_1\|_2 \|S_{W,\varepsilon} f_2\|_2$$

having defined the square function

$$S_{W,\varepsilon} f^2 = \sum_{Q\in\mathcal{S}} \left\langle \left|(W_Q)^{-\frac12} W^{\frac12}\right| |f|_{\mathbb{R}^n} \right\rangle_{1+\varepsilon,Q}^2 \mathbf{1}_Q.$$

Now, if

$$\varepsilon < 2^{-10} t_W, \qquad t_W := (2^{d+N+1} 1 + [W]_{A_\infty})^{-1}, \qquad p := \frac{1+2t}{(1+\varepsilon)(1+t)} \in (1,2)$$

as a result of the sharp reverse Hölder inequality and of the Carleson embedding theorem there holds

$$\|S_{W,\varepsilon} f\|_2^2 \lesssim \sum_{Q\in S} |Q| \langle |f|_{\mathbb{R}^n}\rangle_{\frac{2}{p},Q}^2 \lesssim (p')^p \| |f|_{\mathbb{R}^n} \|_2^2 \lesssim [W]_{A_\infty} \| |f|_{\mathbb{R}^n} \|_2^2$$

cf. [30, Proof of Lemma 5.2], and a similar argument applies to $S_{V,\varepsilon}$. Therefore, for $\varepsilon < 2^{-10} \min\{t_W, t_V\}$, (6.8) turns into

$$\sup_{\|f_j\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}=1} \sum_{Q\in\mathcal{S}} |Q| \langle V^{\frac12} f_1\rangle_{1+\varepsilon,Q} \langle W^{\frac12} f_2\rangle_{1+\varepsilon,Q} \lesssim \sqrt{[W,V]_{A_2}[W]_{A_\infty}[V]_{A_\infty}}$$

which in turn proves Proposition 6.5. □

Remark 6.6. We may derive a slightly stronger weighted estimate than (6.7) for the non-maximally truncated rough integrals $T_{\Omega,\delta}$, by applying Theorem D in conjunction with the $(1, 1+\varepsilon)$ localized estimates proved in [7]. Namely, the estimate

$$\left\| W^{\frac12} T_{\Omega,\delta}(V^{\frac12} f)\right\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)} \lesssim \sup_{\|g\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}=1} \inf_{\varepsilon>0} \sup_{\mathcal{S}} \frac{1}{\varepsilon} \sum_{Q\in\mathcal{S}} |Q| \langle V^{\frac12} f\rangle_{1,Q} \langle W^{\frac12} g\rangle_{1+\varepsilon,Q}.$$

holds uniformly in $\delta > 0$ for all f of unit norm in $L^2(\mathbb{R}^d;\mathbb{R}^n)$. Repeating the proof of Proposition 6.5 then yields the slightly improved weighted estimate

$$\sup_{\delta>0} \left\| W^{\frac12} T_{\Omega,\delta}(V^{\frac12} f)\right\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)} \lesssim \min\{[W]_{A_\infty}, [V]_{A_\infty}\} \sqrt{[W,V]_{A_2}[W]_{A_\infty}[V]_{A_\infty}}\, \|f\|_{L^2(\mathbb{R}^d;\mathbb{R}^n)}\,.$$

6.7. **Proof of Theorem D.** The proof of Theorem D is formally identical to the argument for the scalar valued case, provided that estimate (3.10) and the definition of E_Q given in (3.11) are replaced by suitable vector valued versions. We begin with the second tool. The proof, which is a minor variation on [9, Lemma 3.3], is given below

Lemma 6.8. *Let* $0 < \eta \le 1$, *Q be a dyadic cube and* $f_j \in L^{p_j}(\mathbb{R}^d;\mathbb{R}^n)$, $j = 1, 2$. *Then the set*

$$E_Q := \bigcup_{j=1}^{2} \Big\{ x \in 3Q : \eta \langle f_j \mathbf{1}_{3Q}\rangle_{p_j,L} \not\subset \langle f_j\rangle_{p_j,3Q} \text{ for some cube } L \subset \mathbb{R}^d \text{ with } x \in L \Big\} \tag{6.9}$$

satisfies $|E_Q| \le C\eta^{\min\{p_1,p_2\}}|Q|$ *for some absolute dimensional constant* C.

Proof. We may assume that $\operatorname{supp} f_j \subset 3Q$. It is certainly enough to estimate the measure of each $j \in \{1, 2\}$ component of E_Q by $C\eta^{p_j}|Q|$, and we do so: we fix j and are thus free to write $f_j = f, p_j = p$. Let $\mathcal{L}_f = \{L \subset \mathbb{R}^d : \eta\langle f\rangle_{p,L} \not\subset \langle f\rangle_{p,3Q}\}$. By usual covering arguments it suffices to show that if $L_1, \dots L_m \in \mathcal{L}_f$ are disjoint then

$$\sum_{\mu=1}^{m} |L_\mu| \le C\eta^p|Q|.$$

Fix such a disjoint collection $L_1, \dots L_m$. Notice that if $A \in \mathrm{GL}(\mathbb{R}^n)$ then $\mathcal{L}_{Af} = \mathcal{L}_f$. By action of $\mathrm{GL}(\mathbb{R}^n)$ we may thus reduce to the case where $\mathcal{E}_{\langle f\rangle_{p,3Q}} = B_1$, and in particular

$$B_1 \subset \langle f\rangle_{p,3Q} \subset B_{\sqrt{n}}. \tag{6.10}$$

By membership of each $L_\mu \in \mathcal{L}_f$, we know that $\eta\langle f\rangle_{p,L_\mu} \not\subset B_1$. A fortiori, there exists $\varphi_\mu \in \Phi_{p'}(L_\mu)$ and a coordinate index $\ell_\mu \in \{1, \dots, n\}$ such that

$$\eta(F_\mu)_\ell > \frac{1}{\sqrt{n}}, \qquad F_\mu := \int_{L_\mu} f\varphi_\mu \frac{\mathrm{d}x}{|L_\mu|}. \tag{6.11}$$

Let $M_\ell = \{\mu \in \{1, \dots, m\} : \ell_\mu = \ell\}$. As $\{1, \dots, m\} = \cup\{M_\ell : \ell = 1, \dots, n\}$ it suffices to show that

$$\frac{1}{|3Q|} \sum_{\mu\in M_\ell} |L_\mu| =: \delta < C\eta^p. \tag{6.12}$$

Using the membership $\varphi_\mu \in \Phi_{p'}(L_\mu)$ for the first inequality and the disjointness of the supports for the second equality

$$1 \ge \frac{1}{\delta} \sum_{\mu\in M_\ell} \int_{L_\mu\cap 3Q} |\varphi_\mu|^{p'} \frac{\mathrm{d}x}{|3Q|} = \int_{3Q} |\varphi|^{p'} \frac{\mathrm{d}x}{|3Q|}, \qquad \varphi := \delta^{-\frac{1}{p'}} \sum_{\mu\in M_\ell} \varphi_\mu \mathbf{1}_{3Q}$$

so that $\varphi \in \Phi_{p'}(3Q)$. In particular, beginning with the right inclusion in (6.10) and using (6.11) in the last inequality

$$\sqrt{n} \geq \int_{3Q} (f\varphi)_\ell \frac{\mathrm{d}x}{|3Q|} = \delta^{-\frac{1}{p'}} \sum_{\mu \in M_\ell} \frac{|L_\mu|}{|3Q|} (F_\mu)_\ell > \frac{1}{\eta\sqrt{n}} \delta^{\frac{1}{p}}$$

which rearranging yields (6.12) with $C = n^p$, thus completing the proof. □

At this point, let $\mathcal{S}_k$ be a collection of pairwise disjoint cubes as in Step 3 of the proof of Theorem C. The elements of the collection $\mathcal{S}_{k+1}$ are defined to be the maximal dyadic cubes L such that the same condition as in (3.11) holds, provided the definition of E_Q therein is replaced with the one in (6.9). By virtue of Lemma 6.8, (3.12) still holds provided η is chosen small enough. And, we still obtain that $\mathcal{S}_{k+1}(Q) = \{L \in \mathcal{S}_{k+1} : L \subset 3Q\}$ is a stopping collection. By definition of $\mathcal{S}_{k+1}$, it must be that

$$\langle f_j \mathbf{1}_{3Q} \rangle_{p_j,K} \subset C \langle f_j \rangle_{p_j,3Q} \tag{6.13}$$

whenever the (not necessarily dyadic) cube K is such that a moderate dilate CK of K contains $2^5 L$ for some $L \in \mathcal{S}_{k+1}(Q)$. Fix Q for a moment and let $A_j = (A_j^{m\mu} : 1 \leq m, \mu \leq n) \in \mathrm{GL}(\mathbb{R}^n)$, $j = 1, 2$ be chosen such that the John ellipsoid of $\langle \widetilde{f_j} \rangle_{3Q,p_j}$ is B_1, or its intersection with a lower dimensional subspace in a degenerate case, and $A_j \widetilde{f_j} := f_j$. It follows from (6.13) that if $2^5 L \subset CK$ then $\langle \widetilde{f_j} \mathbf{1}_{3Q} \rangle_{p_j,K} \subset B_C$. This fact, together with Lemma 6.2 readily yields the estimates

$$\|\widetilde{f_j}\|_{\mathcal{Y}_{p_j}(Q)} \lesssim 1, \qquad j = 1, 2. \tag{6.14}$$

We are ready to obtain a substitute for (3.10). In fact

$$\left| \left\langle T[K]_{t_1}^{t_2 \wedge s_Q} \otimes \mathrm{Id}_{\mathbb{R}^n}(f_1 \mathbf{1}_Q), f_2 \right\rangle \right| \leq |Q| \left| \sum_{m=1}^n Q[K]_{t_1}^{t_2}(f_{1m}, f_{2m}) \right| + \sum_{\substack{L \in \mathcal{S}_{k+1}(Q) \\ L \subset Q}} \left| \left\langle T[K]_{t_1}^{t_2 \wedge s_L} \otimes \mathrm{Id}_{\mathbb{R}^n}(f_1 \mathbf{1}_L), f_2 \right\rangle \right|$$

and by actions of $\mathrm{GL}(\mathbb{R}^n)$, see the proof of [9, Lemma 3.4],

$$\begin{aligned} \left| \sum_{m=1}^n Q[K]_{t_1}^{t_2}(f_{1m}, f_{2m}) \right| &= \left| \sum_{m,\mu_1,\mu_2=1}^n A_1^{m\mu_1} A_2^{m\mu_2} Q[K]_{t_1}^{t_2}(\widetilde{f_{1\mu_1}}, \widetilde{f_{2\mu_2}}) \right| \\ &\lesssim \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} \sup_{\mu_1,\mu_2} |Q[K]_{t_1}^{t_2}(\widetilde{f_{1\mu_1}}, \widetilde{f_{2\mu_2}})| \lesssim \langle f_1 \rangle_{p_1,3Q} \langle f_2 \rangle_{p_2,3Q} \end{aligned}$$

where we also employed (3.8) coupled with (6.14) in the last line. Assembling together the last two displays yields the claimed vector-valued version of (3.10), and finishes the proof of Theorem D.

7. Proof of Theorem B

7.1. **Proof of point 1.** We begin with the proof of the first point. Let $1 < p < \infty$, $w \in A_p$ be given, and recall that $\sigma = w^{-\frac{1}{p-1}}$. Fix $\varepsilon > 0$ to be chosen later. Using Theorem A in conjunction with a direct application of [26, Theorem 1.2], we obtain the estimate

$$|\langle Tf, g \rangle| \leq c_{d,p} \varepsilon^{-1} [v]_{A_r}^{\frac{1}{1+\varepsilon} - \frac{1}{p'}} ([u]_{A_\infty}^{\frac{1}{p}} + [v]_{A_\infty}^{\frac{1}{p'}}) \|f\|_{L^p(w)} \|g\|_{L^{p'}(\sigma)},$$

where

$$r=\Big(\tfrac{(1+\varepsilon)'}{p}\Big)'(\tfrac{p}{1+\varepsilon}-1)+1=p+\tfrac{p(p-2)\varepsilon}{1-(p-1)\varepsilon},\qquad v=\sigma^{\frac{1+\varepsilon}{1+\varepsilon-p'}}=w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}},\qquad u=w^{\frac{1+\varepsilon}{1+\varepsilon-p}}.$$

By definition,

$$\begin{aligned}[v]_{A_r}^{\frac{1}{1+\varepsilon}-\frac{1}{p'}}&=\sup_Q\Big(\frac{1}{|Q|}\int_Q w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}\Big)^{\frac{1}{1+\varepsilon}-\frac{1}{p'}}\Big(\frac{1}{|Q|}\int_Q\sigma^{1+\frac{\varepsilon p}{p-(1+\varepsilon)}}\Big)^{(r-1)(\frac{1}{1+\varepsilon}-\frac{1}{p'})}\\&=\sup_Q\Big(\frac{1}{|Q|}\int_Q w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}\Big)^{\frac{1}{p}\frac{1}{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}}\Big(\frac{1}{|Q|}\int_Q\sigma^{1+\frac{\varepsilon p}{p-(1+\varepsilon)}}\Big)^{\frac{1}{p'}\cdot\frac{1}{1+\frac{\varepsilon p}{p-(1+\varepsilon)}}}.\end{aligned}$$

By the sharp reverse Hölder inequality [15], taking $\varepsilon=\frac{1}{\tau_d\max\{p,p'\}\max\{[w]_{A_\infty},[\sigma]_{A_\infty}\}}$, we can conclude

$$\|T\|_{L^p(w)}\le c_{d,p}[w]_{A_p}^{\frac{1}{p}}([w]_{A_p}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}})\max\{[\sigma]_{A_\infty},[w]_{A_\infty}\}\le c_{d,p}[w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}.$$

7.2. **Proof of point 2.** We move onto the Fefferman-Stein type inequality of the second point. Indeed, let $A(t)=t^{pr/\tilde r}$ and $\bar B(t)=t^{\frac12(\frac{p}{\tilde r}+1)}$, where $1<r<p$ and $\tilde r=\frac{pr-\frac{r-1}{2}}{pr-(r-1)}$. Then

$$\sup_Q\langle w^r\rangle_Q^{\frac{1}{pr}}\|(M_rw)^{-\tilde r/p}\|_{B,Q}^{\frac{1}{\tilde r}}\le\sup_Q\inf_{x\in Q}(M_rw)^{\frac1p}\|(M_rw)^{-\tilde r/p}\|_{B,Q}^{\frac{1}{\tilde r}}\le 1.$$

Let $v=M_rw$. Now we have,

$$\begin{aligned}\tilde r'\sum_{Q\in\mathcal S}|Q|\langle f\rangle_{\tilde r,Q}\langle gw^{\frac1p}\rangle_{\tilde r,Q}&=\tilde r'\sum_{Q\in\mathcal S}\langle f^{\tilde r}v^{\frac{\tilde r}{p}}v^{-\frac{\tilde r}{p}}\rangle_Q^{\frac{1}{\tilde r}}\langle gw^{\frac1p}\rangle_{\tilde r,Q}|Q|\\&\le\tilde r'\sum_{Q\in\mathcal S}\|f^{\tilde r}v^{\frac{\tilde r}{p}}\|_{\bar B,Q}^{\frac{1}{\tilde r}}\|v^{-\frac{\tilde r}{p}}\|_{B,Q}^{\frac{1}{\tilde r}}\|w^{\frac{\tilde r}{p}}\|_{A,Q}^{\frac{1}{\tilde r}}\|g^{\tilde r}\|_{\bar A,Q}^{\frac{1}{\tilde r}}|Q|\\&\le 2\tilde r'\sum_{Q\in\mathcal S}\|f^{\tilde r}v^{\frac{\tilde r}{p}}\|_{\bar B,Q}^{\frac{1}{\tilde r}}\|g^{\tilde r}\|_{\bar A,Q}^{\frac{1}{\tilde r}}|E_Q|\\&\le 2\tilde r'\int M_{\bar B}^{\mathcal D}(f^{\tilde r}v^{\frac{\tilde r}{p}})^{\frac{1}{\tilde r}}M_{\bar A(t^{\tilde r})}^{\mathcal D}(g)\\&\le 2\tilde r'\|M_{\bar B}^{\mathcal D}(f^{\tilde r}v^{\frac{\tilde r}{p}})^{\frac{1}{\tilde r}}\|_{L^p}\|M_{\bar A(t^{\tilde r})}^{\mathcal D}(g)\|_{L^{p'}}\\&\le c_dp^2(p')^{\frac1p}(r')^{1+\frac{1}{p'}}\|f\|_{L^p(v)}\|g\|_{L^{p'}}.\end{aligned}$$

Relying upon the sparse domination estimate of Theorem A for $1+\varepsilon=\tilde r$, and duality, we finally reach the bound

$$\|T(f)\|_{L^p(w)}\le c_dp^2(p')^{\frac1p}(r')^{1+\frac{1}{p'}}\|f\|_{L^p(M_rw)}$$

which completes the proof of the second point.

7.3. **Proof of point 3.** Notice that the A_1-A_∞ estimate just follows from the sharp reverse Hölder inequality, so that we may restrict to the case $q>1$. The idea is again to interpret the A_q condition as a bumped A_p condition (see [25, p. 907]). Let $C(t)=t^{\frac{p}{r(q-1)}}$. We have

$$r'\sum_{Q\in\mathcal S}|Q|\langle f\rangle_{r,Q}\langle gw\rangle_{r,Q}\le r'\sum_{Q\in\mathcal S}|Q|\langle f^rw^{\frac rp}\rangle_{\bar C,Q}^{\frac1r}\langle w^{-\frac rp}\rangle_{C,Q}^{\frac1r}\langle g^{rs}w\rangle_Q^{\frac{1}{rs}}\langle w^{(r-\frac1s)s'}\rangle_Q^{\frac{1}{rs'}}.$$

Take

$$r = 1 + \frac{1}{8p(\frac{p}{q})'\tau_d[w]_{A_\infty}}, \qquad s = 1 + \frac{1}{4(\frac{p}{q})'p}.$$

Then $rs < 1 + \frac{1}{2p} < p'$, $r < 1 + \frac{1}{8}(\frac{p}{q} - 1) < \frac{p}{q}$ and $(r - \frac{1}{s})s' < 1 + \frac{1}{\tau_d[w]_{A_\infty}}$. Then applying the sparse domination bound of Theorem A, and the sharp reverse Hölder inequality as in the proof of the first point, we obtain

$$\begin{aligned}
\|T(f)\|_{L^p(w)} &\leq \sup_{\|g\|_{L^{p'}(w)}=1} r' \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_{r,Q}\langle gw\rangle_{r,Q} \\
&\leq \sup_{\|g\|_{L^{p'}(w)}=1} c_{d,p,q}[w]_{A_\infty} \sum_{Q\in\mathcal{S}} |Q|\langle f^r w^{\frac{r}{p}}\rangle_{\tilde{C},Q}^{\frac{1}{r}}\langle g\rangle_{rs,Q}^{w}\langle w\rangle_Q\langle w^{-\frac{1}{q-1}}\rangle_Q^{\frac{q-1}{p}} \\
&\leq \sup_{\|g\|_{L^{p'}(w)}=1} c_{d,p,q}[w]_{A_\infty}[w]_{A_q}^{\frac{1}{p}} \sum_{Q\in\mathcal{S}} \langle f^r w^{\frac{r}{p}}\rangle_{\tilde{C},Q}^{\frac{1}{r}}\langle g\rangle_{rs,Q}^{w} w(Q)^{\frac{1}{p'}}|Q|^{\frac{1}{p}} \\
&\leq \sup_{\|g\|_{L^{p'}(w)}=1} c_{d,p,q}[w]_{A_\infty}[w]_{A_q}^{\frac{1}{p}} \Big(\sum_{Q\in\mathcal{S}} \langle f^r w^{\frac{r}{p}}\rangle_{\tilde{C},Q}^{\frac{1}{r}}|Q|\Big)^{\frac{1}{p}}\Big(\sum_{Q\in\mathcal{S}}(\langle g\rangle_{rs,Q}^{w})^{p'}w(Q)\Big)^{\frac{1}{p'}} \\
&\leq c_{d,p,q}[w]_{A_\infty}^{1+\frac{1}{p'}}[w]_{A_q}^{\frac{1}{p}}\|f\|_{L^p(w)},
\end{aligned}$$

where in the last step we have used the Carleson embedding theorem; we omit the routine details. The proof of point 3. is thus complete.

7.4. **Proof of point 3.** Finally, we prove the Coifman-Fefferman type inequality. Fix $\varepsilon > 0$ and denote $\eta = 1 + \varepsilon$. Also let

$$r = 1 + \frac{1}{8p\eta'\tau_d[w]_{A_\infty}}, \qquad s = 1 + \frac{1}{4\eta' p}.$$

Then again $rs < 1 + \frac{1}{2p} < p'$, $r < \eta$ and $(r - \frac{1}{s})s' < 1 + \frac{1}{\tau_d[w]_{A_\infty}}$. Applying the sparse domination estimate again, we obtain

$$\begin{aligned}
\|T(f)\|_{L^p(w)} &\leq \sup_{\|g\|_{L^{p'}(w)}=1} r' \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_{r,Q}\langle gw\rangle_{r,Q} \lesssim \sup_{\|g\|_{L^{p'}(w)}=1} \eta'[w]_{A_\infty} \sum_{Q\in\mathcal{S}} \langle f\rangle_{\eta,Q}\langle g\rangle_{rs,Q}^{w} w(Q) \\
&\lesssim \sup_{\|g\|_{L^{p'}(w)}=1} \eta'[w]_{A_\infty}^2 \int_{\mathbb{R}^d} M_\eta f M_{rs,w}^{\mathcal{D}}(g) w dx \lesssim \eta'[w]_{A_\infty}^2 \|M_\eta f\|_{L^p(w)},
\end{aligned}$$

which is the estimate we were seeking for. The proof of Theorem B is finally complete.

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